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Analysis on Asymmetric Tail Dependence of Portfolio Returns

포트폴리오 수익률의 비대칭 꼬리분포 의존성 분석

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We use the generalized Pareto distribution and the copula to analyze the impact of asymmetric tail dependence on the risk measures. Simulation results show that the risk measures with symmetric tail dependence underestimate those with asymmetric tail dependence. We also quantify the superiority in the portfolio returns from characterizing the asymmetric tail dependence. The returns of the optimal portfolios from the asymmetric marginals are higher than the returns of the optimal portfolios from the symmetric marginals in the majority of the cases. A caution needs to be exercised in concluding that characterizing the asymmetric dependence in the process of modeling the marginal distributions seems to have an impact on the performance of the optimal portfolio due to the statistical significance of the rate differentials.

Key words: Asymmetric Tail Dependence, Skewness, Value-at-Risk, Expected Shortfall, Generalized Pareto Distribution, Asymmetric Distribution
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I. Introduction

The mean-variance framework assumes symmetric marginal distributions and symmetric dependence between marginal distributions. Departures from symmetricity of the multivariate distribution of portfolio returns have impacts on optimal portfolio choice and risk management. The Arrow-Pratt's risk aversion of investors intuitively allude to the existence of conditional skewness and asymmetric dependence structures. Against a backdrop of these assertions, a line of research has been devoted to analyzing the impact of conditional skewness on optimal portfolio choice. Harvey and Siddique (2000) show that non-elliptical reciprocity of the joint distribution at both extremes would render the discount factor in the pricing kernel to be nonlinear. Harvey and Siddique (1999) suggest controlling for skewness to address time-varying conditional volatility and asymmetric dependence at both ends. The contribution due to Harvey and Siddique (1999) enlightens the importance of conditional skewness in analyzing optimal portfolio choice.

Recent advancement has been made in modeling conditional skewness and asymmetric dependence at the same time. It is proved to be convenient to employ the copula methodology in describing the interdependence structures of the portfolio independently from the marginals of the individual assets. Patton (2004) shows that the mean-variance framework for portfolio selection would not work when the multivariate normality assumption is at odds by the data. Patton (2004) analyzes the asset allocations decisions using copulas when there are asymmetries in the tail area dependence. Patton (2004) analyzes portfolio performance by comparing the realized returns of the portfolio from a bivariate normal copula and the other from a time-varying copula distribution. The gains from competing portfolios are measured by the rate of return differentials between the portfolio strategies. Zhu and Galbraith (2010) employ the asymmetric Student t dynamic conditional score model to control for the asymmetric dependence in the tails of the distributions. The convenience of this approach is that deviations from non-skewness of the multivariate distributions are often taken care of in the process of modeling asymmetric tail dependence. de Roon and Karehnke (2017) develop a skewed distribution from a mixture of two normal distributions and show how skewness has an impact on risk metrics and portfolio choice. This approach evaluates the risk measures based on the calibrated analytical formulas from the normal density. While the procedure due to de Roon and Karehnke (2017) to estimate the risk measures and solve for the optimal portfolio weight from the expected utility maximization is intuitive, their methodology has some restrictive features in it. That is, the skewed distribution in their work is created by combining two normal distributions and is sometimes intractable due to non-invertibility of the normal distribution. Thiele (2020) presents a parametric model based on the asymmetric Student t distribution due to Zhu and Galbraith (2010) combined with the dynamic conditional score procedure. The asymmetric Student t dynamic conditional score model is similar to the skewed t distribution due to Hansen (1994) in the sense that a shape parameter governs skewness. However, the asymmetric Student t distribution is capable of controlling for different thickness between the left and right tail. Armed with this new approach, Thiele (2020) analyzes the impact of asymmetric tail dependence on optimal portfolio choice. Also, the economic gains from modeling asymmetric tail dependence with the asymmetric Student t distribution are numerically approximated.

In this paper, we combine marginal distributions for individual assets and copula distributions for dependence structure to form multivariate distribution for market index portfolio returns. The marginal distributions are designed to represent the difference in tail thickness. The copula distributions are chosen to reflect asymmetric tail dependence. The aim of this paper is two folds. First, we use the generalized Pareto distribution and copula distribution to present the impact of asymmetric tail dependence and fat-tailed behavior on the risk measures. Second, we quantify economic gains from modeling asymmetric tail dependence of the market index returns. We evaluate the rate of return differentials between the optimal portfolios from the asymmetric Student t and asymmetric t distributions against the skewed t distribution.

The remainder of the paper proceeds as follows. Section 2 briefly describes the asymmetric Student t model introduced by Thiele (2020). Section 3 presents our empirical results. Section 4 concludes the discussion.

II. Modeling

We employ a conditional distribution with the asymmetric Student t-distributed residuals for the market index returns:

$$y_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad t = 1, 2, \cdots T, \tag{1}$$
$$z_t \sim (\alpha, \nu_L, \nu_R),$$

where μ is the mean, σ_t is the time-varying standard deviation, and z_t is distributed to asymmetric Student t with shape parameters, α, ν_L, ν_R. The parametric model for the asymmetric Student t-distributed market index returns is:

$$f(y_t : \mu, \sigma_t, \alpha, \nu_L, \nu_R)$$

$$=\begin{cases}\dfrac{1}{\sigma_t}M(\alpha,\nu_L,\nu_R)\left[1+\dfrac{1}{\nu_L}\left(\dfrac{y_t-\mu}{2\alpha^*\sigma_t}\right)^2\right]^{\frac{\nu_L+1}{2}} & y_t \le \mu \\ \dfrac{1}{\sigma_t}M(\alpha,\nu_L,\nu_R)\left[1+\dfrac{1}{\nu_R}\left(\dfrac{y_t-\mu}{2(1-\alpha^*)\sigma_t}\right)^2\right]^{\frac{\nu_R+1}{2}} & y_t > \mu\end{cases}, \quad (2)$$

where

$$M(\alpha,\nu_L,\nu_R)=\frac{\alpha}{\alpha^*}K(\nu_L)=\frac{1-\alpha}{1-\alpha^*}K(\nu_R)K(\nu_i)=\frac{\Gamma\left(\frac{\nu_i+1}{2}\right)}{\Gamma\left(\frac{\nu_i}{2}\right)\sqrt{\pi\nu_i}}$$

$$\alpha^*=\frac{\alpha K(\nu_L)}{\alpha K(\nu_L)+(1-\alpha)K(\nu_R)}.$$

While the parameter α governs the skewness of the distribution, the parameters ν_L and ν_R represent the thickness of the left and right tail, respectively. Therefore, the function M separates out non-symmetry of the distribution into skewness and the difference in the tail-thickness on each side. The skewness parameter α of the asymmetric Student t distribution does not suffice in explaining the difference in the tail-thickness of the distribution, because it can be shown that the tails on each extreme still fade down at the same rate with $\alpha \neq 0.5$. So, irrespective of the value of α, we can parameterize the different rates of decay in the left and right tails of the asymmetric Student t distribution using the parameters ν_L and ν_R. Zhu and Galbraith (2010) show that $\alpha \neq 0.5$ can be derived from the symmetric Student t distribution, with $\nu_L=\nu_R$.

Thiele (2020) applies the first-order dynamic conditional score procedure in order to accommodate a time-varying volatility on the asymmetric Student t distribution in equation (2). To that end, we characterize λ_t from the transformation function, $\sigma_t = \exp(\lambda_t)$ to insure non-negative time-varying standard deviation as the following dynamic equation:

$$\lambda_t = \delta + \phi\lambda_{t-1} + \kappa s_{t-1} \tag{3}$$

$$s_t = H_t u_t = \left[E_{t-1}\left(-\frac{\partial^2 \ln L_t}{\partial \lambda_t^2}\right)\right]^{-1} \frac{\partial \ln L_t}{\partial \lambda_t}, \tag{4}$$

where $u_t = \frac{\partial \ln L_t}{\partial \lambda_t}$ denotes the contribution of the time-varying volatility to the log-likelihood function, and $H_t = \left[E_{t-1}\left(-\frac{\partial^2 \ln L_t}{\partial \lambda_t^2}\right)\right]^{-1}$ is the inverse of the information matrix to rescale the innovations. u_t and H_t can be expressed as follows:

$$u_t = \begin{cases} (\nu_L+1)\dfrac{\frac{1}{\nu_L}\left(\frac{y_t-\mu}{2\alpha^*\sigma_t}\right)^2}{1+\frac{1}{\nu_L}}\left(\dfrac{y_t-\mu}{2\alpha^*\sigma_t}\right)^2 - 1 & y_t \leq \mu \\ (\nu_R+1)\dfrac{\frac{1}{\nu_R}\left(\frac{y_t-\mu}{2(1-\alpha^*)\sigma_t}\right)^2}{1+\frac{1}{\nu_R}}\left(\dfrac{y_t-\mu}{1(1-\alpha^*)\sigma_t}\right)^2 - 1 & y_t > \mu. \end{cases} \tag{5}$$

$$H_t = \begin{cases} \dfrac{\nu_L+3}{2\nu_L} & y_t \leq \mu \\ \dfrac{\nu_R+3}{2\nu_R} & y_t > \mu \end{cases} \tag{6}$$

The equation (5) specifies how the squared deviations from the mean, $\left(\frac{y_t-\mu}{2\alpha^*\sigma_t}\right)^2$ have an impact on the volatility. The structure of equation (5) shows that the thinner the tails of the distribution(ν_L, ν_R get bigger), the bigger the u_t in equation (5) and s_t in equation (4), respectively. So, when the tails of the distribution are very thin, the occurrence of the extreme value of y_t is very rare and gets bigger weight in calculating the time-varying volatility in the next period.

The volatility dynamics in equation (3) can accommodate leverage effects by adding the term, $\kappa^{*} sign(-y_{t-1})(s_{t-1}+1)$, $\kappa^{*} > 0$ in equation (7):

$$\lambda_t = \delta + \phi\lambda_{t-1} + \kappa s_{t-1} + \kappa^{*} sign(-y_{t-1})(s_{t-1}+1), \tag{7}$$

where $\kappa^{*} > 0$. So, when there is a negative return in period t-1 (i.e. $y_{t-1} < 0$), the time-varying volatility gets bigger in period t. Equations (1) - (7) completes the asymmetric Student t distribution dynamic conditional score model with leverage effect.

Ⅲ. Estimation

1. Data

The data for Korea (Korea Composite Stock Price Index, KOSPI), U.S. (S&P 500 Index, S&P 500), Hong Kong (Hang Seng Index, HIS), UK (Financial Times Stock Exchange 100, FTSE 100), China (Shanghai Composite Index, SCI), and Japan (Nikkei 225, NK225) are collected. The daily stock market index return data consists of 2,998 observations and the data spans from January 4, 2007 to July 9, 2021. According to Table 1, all indices' returns are leptokurtic, and the five market indices except for the Hang Seng Index returns are negatively skewed.

〈Table 1〉 Descriptive Statistics

	min	1st quantile	median	3rd quantile	max
KOSPI	-11.64	-0.54	0.07	0.68	11.24
S&P 500	-13.47	-0.42	0.08	0.63	10.33
Hang Seng	-14.70	-0.71	0.04	0.80	16.80
FTSE 100	-10.14	-0.56	0.03	0.63	10.46
Shanghai	-12.83	-0.72	0.05	0.80	14.19
Nikkei 225	-12.92	-0.71	0.06	0.85	13.23

	mean	Standard dev.	skewness	kurtosis
KOSPI	0.03	1.39	-0.59	13.40
S&P 500	0.04	1.42	-0.75	16.20
Hang Seng	0.01	1.68	0.15	15.79
FTSE 100	0.00	1.32	-0.17	12.17
Shanghai	0.01	1.80	-0.47	9.48
Nikkei 225	0.02	1.67	-0.61	11.76

Note: The data for Korea (Korea Composite Stock Price Index, KOSPI), U.S. (S&P 500 Index, S&P 500), Hong Kong (Hang Seng Index, HSI), UK (Financial Times Stock Exchange 100, FTSE 100), China (Shanghai Composite Index, SCI), and Japan (Nikkei 225, NK225) are used for the descriptive statistics. The daily stock market index data consists of 2,998 observations and the data spans from January 4, 2007 to July 9, 2021. According to Table 1, all indices' returns are leptokurtic, and the five market indices except for the Hang Seng Index returns are negatively skewed.

The conjecture that negative shocks induce more volatility than positive shocks can be substantiated by the parameter estimates of the EGARCH model. From the EGARCH model of the form $h_t = a_0 + \sum_{i=1}^{p} a_i \frac{|\epsilon_{t-i}| + \gamma_i \epsilon_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^{p} b_j h_{t-j}$, where $h_t = \log\sigma_t^2$, the leverage effect coefficient γ_i is estimated at -0.366 with the p-value of 0.000 for the KOSPI. For other indices, the leverage effect coefficient γ_i is estimated at -0.722 (p-value=0.000), -0.402 (0.000), -0.999 (0.000), -0.011 (0.690), and -0.556 (0.000) for the S&P 500, Hang Seng Index, FTSE 100, Shanghai Composite Index, and the Nikkei 225, respectively. Except for the Shanghai Composite Index returns, the leverage effects are prevalent in the global stock market index returns.

2. Tail Asymmetry

We report the parameter estimates of the four models, the asymmetric Student t dynamic conditional score, asymmetric t dynamic conditional score, skewed t dynamic conditional score, and t dynamic conditional score models with the leverage effect in Table 2. We use the maximum likelihood procedure and the Newton-Raphson algorithm for the numerical maximization of the Hessian matrix in computing the covariance matrix of the model parameters. The log-likelihood values ($\ln L$) are reported in the last column, and the standard errors for the parameter estimates are in parentheses. Using t-statistics from the parameter estimates of κ^* and the standard errors in parentheses reported in Table 2, the null hypotheses of no leverage effect ($\kappa^* = 0$) in the conditional volatility are rejected at a 1% level of significance for the asymmetric Student t, asymmetric t, skewed t, and the t distribution models except for the SCI. It is apparent that the conditional volatilities are highly persistent from the parameter estimate ϕ in equation (3) for all six indices and the autocorrelation functions in Figure 1. Table 3 reports the Bayesian information criterion for the asymmetric Student t, asymmetric t, skewed t, and t models using the six indices. For the five market indices, the Bayesian information criteria are minimized when the asymmetric Student t distributions are used. The BIC for the Hang Seng Index is minimized with the skewed t distribution, however, the BICs are almost identical for the asymmetric Student t and the skewed t distributions. The parameter estimates of α from the asymmetric Student t model are reported in Table 2. The parameter estimates of α for the KOSPI and SCI are 0.5, and those for the S&P 500, HSI, FTSE 100, and the NK225 are close to 0.5. For all indices, removing skewness is evidenced from the statistically significant parameter estimates of $\alpha = 0.5$.

Judging from the parameter estimates of ν_L and ν_R, all market indices returns have thicker left tails and thinner right tails than the normal. From the estimation results, we can conclude that the asymmetric Student t distribution is capable of controlling for different thickness between the left and right tail. Since the skewed t and the t distributions are assumed to have the symmetric tails, on the other hand, those two models are mis-specified.

〈Table 2〉 Model Estimation Results

KOSPI	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	-0.001 (0.001)	0.991 (0.003)	0.053 (0.006)	0.500 (0.013)	0.026 (0.004)	3.577 (0.431)	9.011 (1.279)	0.062 (0.032)	4,426
Asymmetric t	-0.002 (0.001)	0.988 (0.004)	0.057 (0.007)	-	0.027 (0.004)	3.612 (0.352)	8.896 (0.604)	0.053 (0.004)	4,426
Skewed t	-0.003 (0.001)	0.989 (0.004)	0.057 (0.007)	0.536 (0.011)	0.029 (0.005)	5.126 (0.498)	-	0.127 (0.027)	4,434
t	-0.004 (0.001)	0.990 (0.002)	0.052 (0.007)	-	0.035 (0.004)	12.751 (2.227)	-	-0.033 (0.010)	4,440

S&P 500	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	-0.009 (0.003)	0.971 (0.005)	0.085 (0.009)	0.515 (0.012)	0.075 (0.006)	3.541 (0.321)	9.501 (2.104)	0.108 (0.023)	4,125
Asymmetric t	-0.008 (0.003)	0.968 (0.004)	0.085 (0.009)	-	0.074 (0.006)	3.377 (0.279)	10.814 (0.848)	0.082 (0.013)	4,134
Skewed t	-0.011 (0.003)	0.973 (0.005)	0.085 (0.008)	0.541 (0.010)	0.076 (0.007)	4.782 (0.403)	-	0.145 (0.021)	4,112
t	-0.012 (0.003)	0.967 (0.005)	0.085 (0.008)	-	0.075 (0.007)	4.504 (0.357)	-	0.078 (0.013)	4,111

HSI	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	0.001 (0.001)	0.986 (0.004)	0.051 (0.007)	0.515 (0.014)	0.025 (0.004)	5.033 (0.694)	6.618 (1.128)	0.084 (0.042)	5,114
Asymmetric t	0.001 (0.001)	0.985 (0.004)	0.051 (0.007	-	0.025 (0.004)	4.673 (0.545)	7.204 (0.418)	0.050 (0.021)	5,115
Skewed t	0.001 (0.001)	0.987 (0.004)	0.051 (0.007)	0.528 (0.011)	0.026 (0.005)	5.682 (0.587)	-	0.123 (0.036)	5,115
t	0.001 (0.001)	0.985 (0.004)	0.059 (0.007)	-	0.025 (0.005)	5.607 (0.571)	-	0.044 (0.021)	5,118

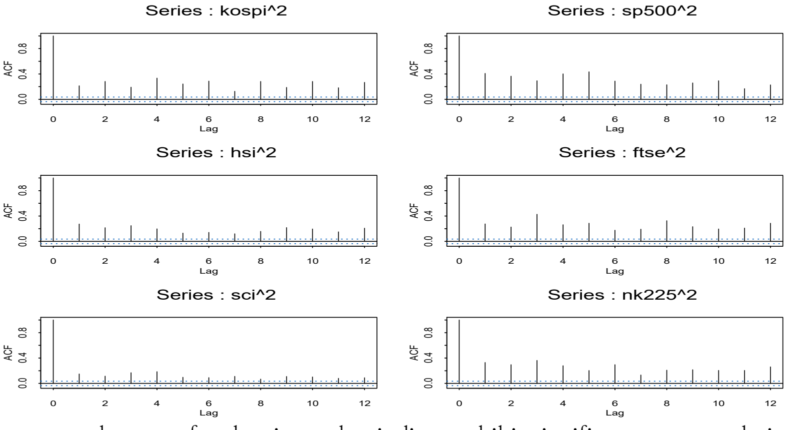
FTSE 100	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	-0.003 (0.002)	0.983 (0.004)	0.047 (0.007)	0.506 (0.014)	0.054 (0.005)	4.386 (0.522)	8.740 (1.893)	0.033 (0.032)	4,326
Asymmetric t	-0.003 (0.002)	0.982 (0.004)	0.047 (0.007)	-	0.054 (0.005)	4.271 (0.443)	9.084 (0.637)	0.025 (0.015)	4,327
Skewed t	-0.005 (0.001)	0.986 (0.004)	0.047 (0.007)	0.535 (0.010)	0.055 (0.005)	5.608 (0.548)	-	0.092 (0.026)	4,330
t	-0.005 (0.002)	0.981 (0.004)	0.046 (0.007)	-	0.054 (0.005)	5.503 (0.526)	-	0.020 (0.015)	4,336

SCI	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	0.001 (0.001)	0.999 (0.001)	0.049 (0.005)	0.500 (0.013)	0.002 (0.003)	3.441 (0.398)	4.617 (0.588)	0.040 (0.035)	5,291
Asymmetric t	0.001 (0.001)	0.999 (0.001)	0.049 (0.006)	-	0.003 (0.003)	3.447 (0.358)	4.617 (0.198)	0.039 (0.000)	5,291
Skewed t	0.001 (0.002)	0.999 (0.004)	0.049 (0.007)	0.512 (0.010)	0.002 (0.004)	3.929 (0.309)	-	0.068 (0.029)	5,293
t	0.001 (0.002)	0.999 (0.004)	0.049 (0.007)	-	0.002 (0.004)	3.906 (0.302)	-	0.040 (0.019)	5,294

NK225	δ	ϕ	κ	α	κ^*	ν_L	ν_R	μ	lnL
Asymmetric Student t	0.005 (0.003)	0.962 (0.007)	0.076 (0.008)	0.512 (0.014)	0.052 (0.006)	4.895 (0.608)	8.830 (1.904)	0.085 (0.047)	5,165
Asymmetric t	0.005 (0.003)	0.962 (0.008)	0.075 (0.009)	-	0.051 (0.006)	4.666 (0.515)	9.726 (0.738)	0.055 (0.021)	5,165
Skewed t	0.003 (0.002)	0.964 (0.008)	0.077 (0.009)	0.530 (0.011)	0.052 (0.006)	6.119 (0.655)	-	0.131 (0.037)	5,169
t	0.004 (0.002)	0.962 (0.008)	0.074 (0.009)	-	0.051 (0.006)	5.919 (0.608)	-	0.046 (0.021)	5,173

Note: We report the parameter estimates of the four models with leverage effect and the log-likelihood values ($\ln L$) in the table. The standard errors are in parentheses. As reported in the sixth column, the null hypothesis of no leverage effect ($\kappa^* = 0$) in the conditional volatility is rejected at a 1% level of significance for all models except for the SCI. To reflect the persistency and leverage effect of the conditional volatility, we estimate the asymmetric Student t, asymmetric t, skewed t, and the t distribution models. The parameter estimates of α for the KOSPI and SCI are 0.5, and those for the S&P 500, HSI, FTSE 100, and the NK225 are close to 0.5. For all indices, removing skewness is evidenced from the statistically significant parameter estimates of $\alpha = 0.5$. As the parameter estimates of ν_L and ν_R show, all market indices returns have thicker left tails and thinner right tails than the normal. The estimation results in the table show that the asymmetric Student t distribution is capable of controlling for different thickness between the left and right tail.

〈Figure 1〉 Autocorrelation Functions of Squared Index Returns



Note: The squared returns for the six market indices exhibit significant autocorrelation. This indicates that the conditional volatilities are persistent. It is apparent that the conditional volatilities are highly persistent from the parameter estimate ϕ for all six indices reported in Table 2 and the autocorrelation functions in the above figures.

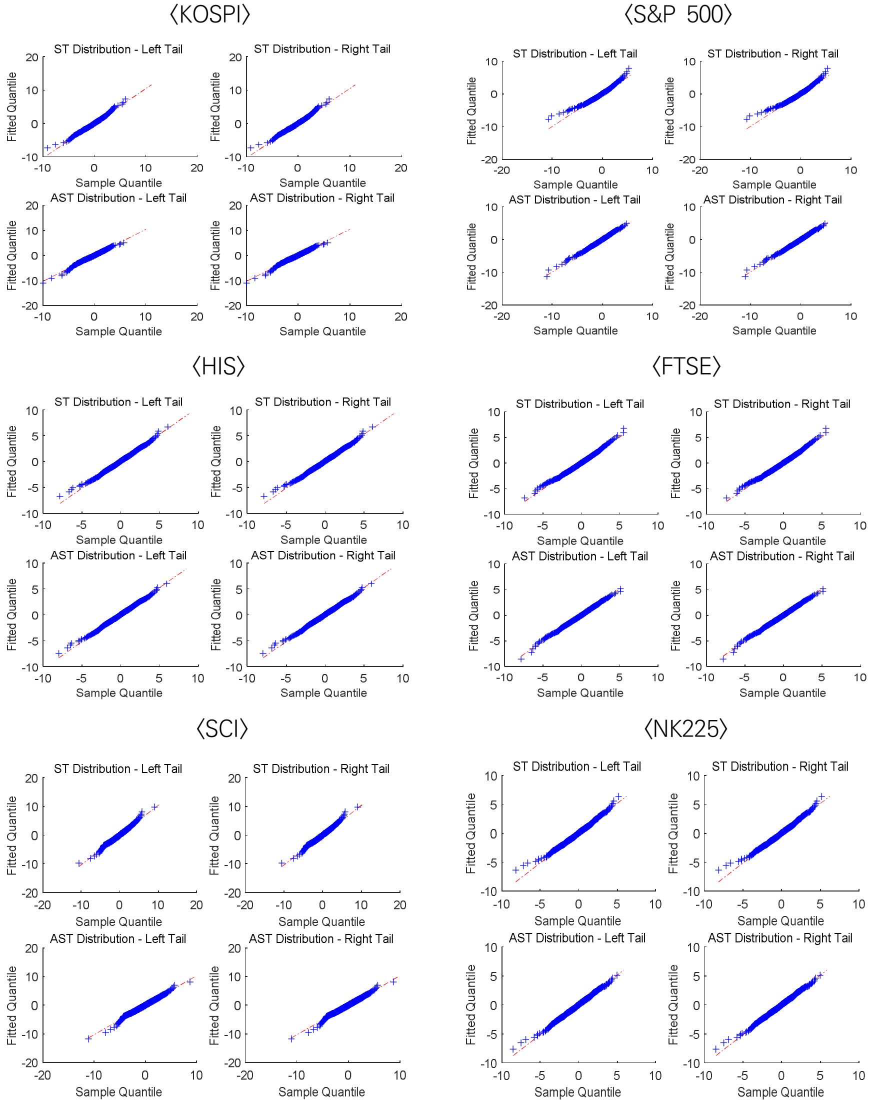
〈Table 3〉 Model Selection Criterion

	KOSPI	S&P	HSI	FTSE	SCI	NK225
Asymmetric Student t	8,916.3	8,286.6	10,293	8,716.6	10,647	10,396
Asymmetric t	**8,907.4**	**8,278.1**	10,287	**8,709.0**	**10,639**	**10,388**
Skewed t	8,924.3	8,306.9	**10,286**	8,716.1	10,642	10,395
t	9,247.2	8,961.5	10,571	9,024.2	10,931	-

Note: The table reports the Bayesian information criterion for the asymmetric Student t dynamic conditional score, asymmetric t, skewed t, and t models using the six market indices. The tail asymmetries of the data are best captured by the asymmetric t model except for the Hong Kong Hang Seng Index. For the five market indices, removing skewness is evidenced from the statistically significant parameter estimates of $\alpha = 0.5$.

Figure 2 shows qq-plots for the standardized residuals from the skewed t dynamic conditional score and asymmetric Student t dynamic conditional score models against a reference distribution with symmetric tails. Most of the qq-plots from the asymmetric Student t dynamic conditional score model do not deviate much from a reference distribution. However, the qq-plot from the asymmetric Student t dynamic conditional score model for the Shanghai Composite Index returns suggests that the tails of the asymmetric Student t dynamic conditional score model are still fatter than the tails of the symmetric-tailed distribution.

〈Figure 2〉 QQ Plots against Asymmetric Student t and Skewed t Reference Distributions



Note: The figure shows qq-plots for the standardized residuals from the estimated skewed t dynamic conditional score and asymmetric Student t dynamic conditional score models against a reference distribution with symmetric tails. For each index, the upper panel shows the qq-plots of the standardized residuals from the estimated skewed t dynamic conditional score model and the lower panel graphs the qq-plots of the standardized residuals from the estimated asymmetric Student t model. The qq-plot from the asymmetric Student t dynamic conditional score model for the Shanghai Composite Index returns suggests that the tails of the asymmetric Student t dynamic conditional score model are still fatter than the tails of the symmetric-tailed distribution. For other indices, the qq-plots prove that the asymmetric Student t dynamic conditional score model estimation addresses the fat-tailed issues.

3. Risk Measures

In this section, we analyze the impact of tail shape on the risk measures. If the tails of the marginal distributions of the market index returns are thicker than normal, we would underestimate the risk measures by assuming that the marginal distributions are normally distributed. We use the generalized Pareto distribution, GPD henceforth, as the fully parametric model for the tails of the marginal distribution.

Let $X_1, X_2, \ldots$ be independently identically distributed losses with an unknown CDF F. Then, the conditional probability of the excesses over a threshold u can be expressed as follows:

$$F_u(y) = \Pr\{X - u \leq y | X > u\} = \frac{F(y+u) - F(u)}{1 - F(u)}, \quad y > 0$$

Embrechts et al. (1997) show that the generalized Pareto distribution is the close approximation to the above excess distribution with a positive function $\beta(u)$.

$$G_{\xi,\beta(u)}(y) = \begin{Bmatrix} 1 - \left(1 + \frac{\xi y}{\beta(u)}\right)^{-\frac{1}{\xi}} & \text{for} \quad \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta(y)}\right) & \text{for} \quad \xi = 0 \end{Bmatrix}, \quad \beta(u) > 0$$

defined for $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq \beta(u)\xi$ when $\xi < 0$.

The most daunting task in the maximum likelihood estimation of the GPD model is to choose the optimal threshold value where the marginal distribution starts to fade out to the tail area. Goldie and Smith (1987) and Hall (1990), Danielsson and de Vries (1998), Danielsson et al. (2001) choose the optimal threshold value by using a subsample bootstrap procedure.

To show how tail thickness affects the risk measure estimation, however, the sample mean excess function plot due to McNeil and Saladin (1997), McNeil and Frey (2000) and Zivot and Wang (2006) would be sufficient. The bias and the efficiency involved in the estimation of the GPD model are dependent on the number of observations pertained to the tail area and the center of the marginal distribution. There is no clear-cut solution to this problem.

We choose 0.9% and -0.9% as the upper tail and right tail area threshold values and estimate the GPD marginals based on the excesses over threshold values. For the KOSPI, we use 18.26% of the 2,995 observations with the threshold value of 0.9% to estimate the upper tail parameter estimates of the GPD model. Also, we use 16.76% of the observations with the threshold value of -0.9% to estimate the lower tail parameter estimates of the GPD model. The tail shape parameters on both sides with the KOSPI and S&P 500 are reported in Tables 4 and 5. For the other four indices, we do not report the estimation results to save space, however, the results are available upon request.

〈Table 4〉 GPD Model Estimation with the KOSPI

Upper Tail Estimate with the Threshold at 0.9% (18.26% of observations)			
	Value	Standard Error	t-ratio
	0.196	0.049	4.02
	0.709	0.046	15.54
Lower Tail Estimate with the Threshold at -0.9% (16.76% of observations)			
	Value	Standard Error	t-ratio
	0.224	0.055	4.06
	0.854	0.060	14.23

Note: The GPD model estimation results are reported in the table. We choose 0.9% and -0.9% as the upper tail and right tail area threshold values and estimate the GPD marginals based on the excesses over threshold values. For the KOSPI, we use 18.26% of the 2,995 observations with the threshold value of 0.9% to estimate the upper tail parameter estimates of the GPD model. Also, we use 16.76% of the observations with the threshold value of -0.9% to estimate the lower tail parameter estimates of the GPD model. The tail shape parameters on both sides with the KOSPI are reported in the table. For the other four indices, we do not report the estimation results to save space, however, the results are available upon request.

〈Table 5〉 GPD Model Estimation with the S&P 500

Upper Tail Estimate with the Threshold at 0.9% (17.23% of the data)			
	Value	Standard Error	t-ratio
	0.317	0.058	5.44
	0.636	0.046	13.94
Lower Tail Estimate with the Threshold at -0.9% (14.89% of the data)			
	Value	Standard Error	t-ratio
	0.182	0.054	3.38
	1.010	0.072	14.03

Note: The GPD model estimation results are reported in the table. ut solution to this problem. We choose the same threshold values as the KOSPI. For the S&P 500, we use 17.23% of the 2,995 observations to estimate the upper tail parameter estimates of the GPD model. Also, we use 14.89% of the observations to estimate the lower tail parameter estimates of the GPD model. The tail shape parameters on both sides with the S&P 500 are shown. For the other four indices, we do not report the estimation results to save space, however, the results are available from the author upon request.

From the statistically significant positive estimates of the tail shape parameter ξ, we can determine that the left tails and right tails of the KOSPI and S&P 500 have fat-tailed distributions. The tail shape parameters ξ for the HSI are 0.16 (t-ratio 4.17) and 0.11 (17.51), for the FTSE 100 are 0.24 (4.44) and 0.13 (2.58), for the SCI are 0.06 (1.63) and 1.13 (18.90), and for the NK225 are 0.09 (2.45) and 0.15 (3.55). Therefore, except for the upper tail of the SCI, we can conclude that the upper tails and lower tails of the six market indices returns have fat tails on both sides of the marginal distributions.

We consider fifteen pairs of one-period investment in two market indices. For each portfolio with expected return R and the joint distribution function F_R, we define two risk measures VaR and ES for a given level of loss:

$$VaR_q = F_{-R}^{-1}(R) \tag{8}$$

$$ES_q = E[-R| -R > VaR_q] \tag{9}$$

We generate random simulations from the joint distribution function F_R and calculate numerical approximations to the VaR and ES. The KOSPI daily returns are distributed to the GPD as shown in Table 4 and the S&P 500 daily returns are distributed to the GPD in Table 5. We use the Clayton copula with the correlation parameter θ, $C(u,v;\theta)=\left[u^{-\theta}+v^{-\theta}-1\right]^{-\frac{1}{\theta}}$, where $\theta>0$ that reflects the asymmetric lower tail dependence parameter $\tau^{L}=2^{-\frac{1}{\theta}}$ and the upper tail dependence parameter $\tau^{U}=0$. We also use the Frank copula which does not exhibit lower or upper tail dependence. The rationale behind the choice of the Clayton copula lies in the tail asymmetry, while the Frank copula and Gaussian copula (for the case of correlation coefficient 〈 1) do not exhibit lower or upper tail dependence. In spite of the dependence structure between the marginal distributions of copulas, the Gaussian copula is the most generally used copula. We analyze the impact of the dependence structures of copulas on the risk measures using the Clayton, Gaussian, and Frank copula.

According to Table 6, the portfolio that invests 50% of wealth in KOSPI and 50% in S&P 500 could make -1.25% loss or more in one trading day with 5% probability. If that event occurs, then, the expected loss could be as low as -2.14%. When $q=0.99$, the VaR and ES are calculated as -2.62% and -3.84%. To compare this result with symmetric tail dependence, we use the normal copula. For a given level of loss $q=0.95$, the VaR and ES are calculated as -1.16% and -1.82%. That is, the portfolio that invests 50% of wealth in KOSPI and 50% in S&P 500 could make -1.16% loss or more in one trading day with 5% probability. If the event occurs, then, the expected loss could be as low as -1.82%. When $q=0.99$, the VaR and ES are calculated as -2.22% and -2.98%.

Table 6 also shows numerical approximation to the VaR and ES when the portfolios are composed of a pair of market indices. For a given level of loss $q=0.95$, the VaR ranges from -1.25% to -1.44%, and the ES ranges from -2.54% to -2.91% when the Clayton copula is used to describe the dependence structure of the portfolio returns. The important fact that we can read off from the table is that the risk measures are underestimated when we assume that the dependence structures of the portfolios are characterized by the Frank copula. The Frank copula assumes no lower or upper tail dependence. The Gaussian copula also tends to underestimate the risk measures compared to the Clayton copula. It should be noted from this simulation that modeling the dependence structures of the multivariate portfolio returns is important in evaluating and managing risk.

〈Table 6〉 VaR and ES by Simulation with the GPD Margins and the Parametric Copulas

KOPSI and S&P 500	VaR		ES	
	95%	99%	95%	99%
Gaussian	-1.16	-2.22	-1.82	-2.98
Frank	-1.13	-2.02	-1.69	-2.65
Clayton	-1.25	-2.62	-2.14	-3.84

KOSPI and HSI	VaR		ES	
	95%	99%	95%	99%
Gaussian	-1.32	-2.38	-2.01	-3.19
Frank	-1.13	-2.02	-1.69	-2.65
Clayton	-1.42	-2.84	-2.34	-4.05

KOSPI and FTSE 100	VaR		ES	
	95%	99%	95%	99%
Gaussian	-1.21	-2.20	-1.83	-2.92
Frank	-1.17	-2.01	-1.70	-2.60
Clayton	-1.27	-2.54	-2.10	-3.65

KOSPI and SCI	VaR		ES	
	95%	99%	95%	99%
Gaussian	-1.36	-2.48	-2.07	-3.33
Frank	-1.31	-2.28	-1.93	-3.00
Clayton	-1.44	-2.91	-2.42	-4.30

KOSPI and NK225	VaR		ES	
	95%	99%	95%	99%
Gaussian	-1.33	-2.37	-2.00	-3.17
Frank	-1.30	-2.22	-1.89	-2.88
Clayton	-1.43	-2.82	-2.36	-4.13

Note: We use the Clayton copula and the Frank copula along with the GPD estimation results in Tables 4 and 5 to approximate the VaR and ES. The risk measures with the Gaussian, Frank, and the Clayton copula are reported in each panel. The risk measures are underestimated when we assume that the dependence structures of the portfolios are characterized by the Gaussian copula or the Frank copula. The Frank copula assumes no lower or upper tail dependence. The Gaussian copula also tends to underestimate the risk measures compared to the Clayton copula. Each panel presents the simulation results from the portfolio which consists of a pair of market index, that is, KOSPI and S&P 500, KOSPI and HIS, KOSPI and FTSE 100, KOSPI and SCI, and KOSPI and NK225, respectively. For the rest of the ten pairs of portfolios, we do not present the results to save space.

4. Portfolio Performance Evaluation

The aim in this section is to measure the rate of return differentials that investors can increase by taking the alternative optimal portfolio choice that takes asymmetries and/or skewness of the distribution into account. That is, we quantify the difference in the rate of returns from characterizing the asymmetric tail dependence of the bivariate market index portfolio returns. To the best of our knowledge, Thiele (2020) suggests evaluating the rate of return gains from selecting a portfolio that takes asymmetric tail dependence into account for the first time. Following Thiele's procedure, we generate the realized returns of the optimal portfolios from the asymmetric tail dependence models and the skewed t models and calculate the rate of return differentials between the two models. The estimation window consists of 1,000 trading days and moves forward by five days in each step of the procedure. We repeat the simulations 100,000 times. The investor can choose the optimal portfolios that maximize the expected utility. The optimal portfolios are chosen from the four combinations of strategies. That is, the marginal distributions for individual index returns are assumed to be distributed to the asymmetric Student t and the asymmetric t. The dependence structures of the portfolios are selected from the Frank copula and the Clayton copula. The rate of return differentials are calculated against the skewed t.

The positive rate of return differentials reported in Tables from 7 to 10 mean that the returns of the optimal portfolios of the asymmetric Student t marginals or the asymmetric t marginals coupled with the Frank copula or the Clayton copula are higher than the returns of the optimal portfolios of the skewed t marginals combined with the Frank copula or the Clayton copula. The statistics in boldface are statistically significant at a 10% significance level. According to Tables from 7 to 10, the returns of the optimal portfolios from the asymmetric Student t or the asymmetric t marginals are higher than the returns of the optimal portfolios from the skewed t marginals in most cases irrespective of the copula distribution. In twelve cases out of fifteen from Table 7, for example, the rate of return differentials are positive. However, the positive rate of return differentials are statistically significant only in two cases with boldface numbers. On the other hand, the positive rate of return differentials reported in Table 8 mean that the returns of the optimal portfolios of the asymmetric t marginals and the Frank copula are higher than the returns of the optimal portfolios of the skewed t marginals and the Frank copula. The statistics in boldface are statistically significant at a 10% significance level. In twelve cases, the rate of return differentials are positive. In seven cases with boldface numbers, the positive rate of return differentials are statistically significant. From Table 10, we find four cases of negative rate differentials. The negative rate differentials in Table 10 show that the returns of the optimal portfolios from the asymmetric t marginals and the Clayton copula are lower than the returns of the optimal portfolios from the skewed t marginals and the Clayton copula. However, the negative rate differentials are insignificant. For the rest eleven cases, the returns of the optimal portfolios from the asymmetric t marginals and the Clayton copula are higher than the returns of the optimal portfolios from the skewed t marginals and the Clayton copula. Furthermore, five out of eleven cases, the rate of return differentials are significantly positive. From the empirical results reported in Tables 7-10, we can conclude that characterizing the asymmetric dependence in the process of modeling the marginal distributions and the copula functions seems to have a positive impact on the performance of the optimal portfolio. The positive impacts on the performance of the optimal portfolios are conspicuous when we employ the asymmetric t in characterizing tail asymmetry. This finding is in line with the model selection results in Table 3. We find that the asymmetric t dynamic conditional score model best fits the KOSPI, S&P 500, FTSE 100, SCI, and the NK225 based on the BIC. A word of caution is in order with respect to the empirical results reported in Tables from 7 to 10. There are not many instances that the returns of the optimal portfolios of the asymmetric Student t marginals or the asymmetric t marginals are lower than the returns of the optimal portfolios of the skewed t marginals. However, the positive rate differentials are statistically significant only in seven and four cases in Tables 8 and 10, respectively.

〈Table 7〉 Asymmetric Student t vs. Skewed t with Frank Copula

	S&P 500	HSI	FTSE 100	SCI	NK225
KOSPI	**3.42** (0.03)	-2.22 (0.86)	1.57 (0.23)	0.36 (0.44)	1.97 (0.20)
S&P 500	-	0.27 (0.44)	1.67 (0.19)	**3.42** (0.06)	2.06 (0.13)
HSI	-	-	1.13 (0.17)	1.78 (0.19)	-0.02 (0.51)
FTSE 100	-	-	-	2.84 (0.13)	-0.95 (0.73)
SCI	-	-	-	-	1.34 (0.29)

Note: We generate the realized returns of the optimal portfolios from the asymmetric Student t distribution and the skewed t distribution and calculate the rate of return differentials between the two models. The dependence structures of the portfolios are selected from the Frank copula and the Clayton copula. The estimation window consists of 1,000 trading days and moves forward by five days in each step of the procedure. We repeat the simulations 100,000 times. The positive rate of return differential reported in the table means that the returns of the optimal portfolios of the asymmetric Student t marginals and the Frank copula are higher than the returns of the optimal portfolios of the skewed t marginals and the Frank copula. The statistics in boldface are statistically significant at a 10% significance level. In twelve cases, the rate of return differentials are positive, however, only two cases with boldface numbers, the positive rate of return differentials are statistically significant.

〈Table 8〉 Asymmetric t vs. Skewed t with Frank Copula

	S&P 500	HSI	FTSE 100	SCI	NK225
KOSPI	**6.97** (0.00)	-2.24 (0.85)	**4.58** (0.02)	2.22 (0.18)	1.81 (0.28)
S&P 500	-	1.07 (0.24)	**2.64** (0.04)	**6.55** (0.01)	**2.59** (0.07)
HSI	-	-	1.31 (0.22)	3.35 (0.12)	-0.54 (0.59)
FTSE 100	-	-	-	**5.97** -	-0.52 (0.60)
SCI	-	-	-	-	**4.43** (0.06)

Note: The positive rate of return differentials reported in the table mean that the returns of the optimal portfolios of the asymmetric t marginals and the Frank copula are higher than the returns of the optimal portfolios of the skewed t marginals and the Frank copula. The statistics in boldface are statistically significant at a 10% significance level. In twelve cases, the rate of return differentials are positive. In seven cases with boldface numbers, the positive rate of return differentials are statistically significant.

〈Table 9〉 Asymmetric Student t vs. Skewed t with Clayton Copula

	S&P 500	HSI	FTSE 100	SCI	NK225
KOSPI	2.59 (0.05)	-1.64 (0.91)	0.62 (0.36)	-0.25 (0.56)	0.43 (0.38)
S&P 500	-	-0.17 (0.55)	0.09 (0.47)	**3.49** (0.04)	1.29 (0.14)
HSI	-	-	0.58 (0.25)	0.57 (0.35)	-0.97 (0.80)
FTSE 100	-	-	-	0.28 (0.46)	-1.29 (0.83)
SCI	-	-	-	-	0.05 (0.50)

Note: The positive rate of return differentials reported in the table mean that the returns of the optimal portfolios of the asymmetric Student t marginals and the Clayton copula are higher than the returns of the optimal portfolios of the skewed t marginals and the Clayton copula. The statistics in boldface are statistically significant at a 10% significance level. In ten cases, the rate of return differentials are positive. Only in two cases with boldface numbers, however, the positive rate of return differentials are statistically significant.

〈Table 10〉 Asymmetric t vs. Skewed t with Clayton Copula

	S&P 500	HSI	FTSE 100	SCI	NK225
KOSPI	**5.91** (0.00)	-1.63 (0.88)	**3.05** (0.05)	1.78 (0.19)	-0.07 (0.50)
S&P 500	-	0.55 (0.33)	0.31 (0.40)	**7.01** (0.00)	1.24 (0.22)
HSI	-	-	0.55 (0.33)	1.84 (0.18)	-1.66 (0.82)
FTSE 100	-	-	-	3.46 (0.17)	-0.75 (0.65)
SCI	-	-	-	-	**3.10** (0.09)

Note: The positive rate of return differentials reported in the table mean that the returns of the optimal portfolios of the asymmetric t marginals and the Clayton copula are higher than the returns of the optimal portfolios of the skewed t marginals and the Clayton copula. The statistics in boldface are statistically significant at a 10% significance level. In eleven cases, the rate of return differentials are positive. In four cases with boldface numbers, the positive rate of return differentials are statistically significant.

Ⅳ. Concluding Remarks

In this paper, we use the generalized Pareto distribution and the copula distributions to analyze the impact of asymmetric tail dependence and fat-tailed behavior on the risk measures. We also quantify the difference in the rate of returns between the optimal portfolios generated by the asymmetric Student t distribution and the asymmetric t distribution against the skewed t distribution. The Clayton copula and the Frank copula which exhibit only lower tail dependence and no tail dependence respectively are employed.

We consider fifteen pairs of one-period investment in two market indices. We generate random simulations from the joint distribution function and calculate numerical approximations to the VaR and ES. Simulation results show that the risk measures with symmetric tail dependence underestimate those with asymmetric tail dependence. For the KOSPI returns at a given level of loss $q=0.95$, the VaR and ES with symmetric tail dependence underestimate risk by 7.2% and 15.0%, respectively.

We also quantify the rate of return differentials that investors can get by taking the asymmetries and/or skewness into account in portfolio selection. We generate realized returns of the optimal portfolios and calculate the difference in the rate of returns of the competing portfolios. The optimal portfolios are chosen under the assumption that marginal distributions for individual index returns are assumed to be distributed to the asymmetric Student t and asymmetric t. The dependence structures of the portfolios are selected from the Frank copula and the Clayton copula. The rate of return differentials are calculated against the optimal portfolio based on the skewed t marginal distribution. The returns of the optimal portfolios from the asymmetric Student t marginals (twelve cases in Table 8) or the asymmetric t marginals (eleven cases in Table 10) are higher than the returns of the optimal portfolios from the skewed t marginals irrespective of the copula distribution. There are not many instances (less than five cases at the maximum in each Table) that the returns of the optimal portfolios of the asymmetric Student t marginals or the asymmetric t marginals are lower than the returns of the optimal portfolios of the skewed t marginals. However, the positive rate of return differentials are statistically significant only in seven and four cases in Tables 8 and 10, respectively. It can be important to include the asymmetric tail dependence in the process of characterizing the marginal distributions of asset returns to improve the performance of the optimal portfolio, however, a caution needs to be exercised due to statistical significance.

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요 약

본 연구는 다음과 같은 두 가지 실증분석 결과를 도출하고 있다. 첫째, 일반화 파레토분포와 코퓰라함수를 이용하여 꼬리분포의 비대칭성이 포트폴리오의 위험척도에 미치는 영향을 추정하였다. 시뮬레이션 결과는 대칭적 꼬리분포를 이용하는 위험척도가 실제 위험척도를 과소평가하는 것으로 나타났다. 둘째, 꼬리분포의 비대칭성을 반영한 최적포트폴리오가 꼬리분포의 대칭성을 가정한 최적포트폴리오에 비해 수익률 면에서 우월한 정도를 측정하였다. 실증분석 결과는 포트폴리오의 한계분포와 포트폴리오 구성자산 간 의존성분포를 모형화 함에 있어 꼬리분포의 비대칭성 정보를 포함할 경우 최적포트폴리오의 기대수익률이 상승하는 것으로 나타났다. 꼬리분포의 비대칭성을 모형화하는 것이 최적포트폴리오에 미치는 영향은 수익률 우월성 추정치의 통계적 유의성으로 인해 해석에 유의해야 할 것으로 보인다.

국문색인어: 비대칭꼬리분포, 왜도, 일반화 파레토분포, 기대손실